A RECURRENCE FORMULA SOLVES A THEORETICAL PROBLEM ABOUT THE AXIS OF LEAST INERTIA OF A TWO-DIMENSIONAL OBJECT.

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The axis of least inertia of a two-dimensional object is a fundamental tool in computer vision. In this write-up, we consider a theoretical question (which ultimately may be of practical utility) about the axis of least inertia of a 2-dimensional object. Consider Figure 1. There are n objects in the figure. The problem we consider is to: ***give a procedure for computing the axis of least inertia of the first object, then the first two objects, then the first three objects, then the first four objects, and so on***.

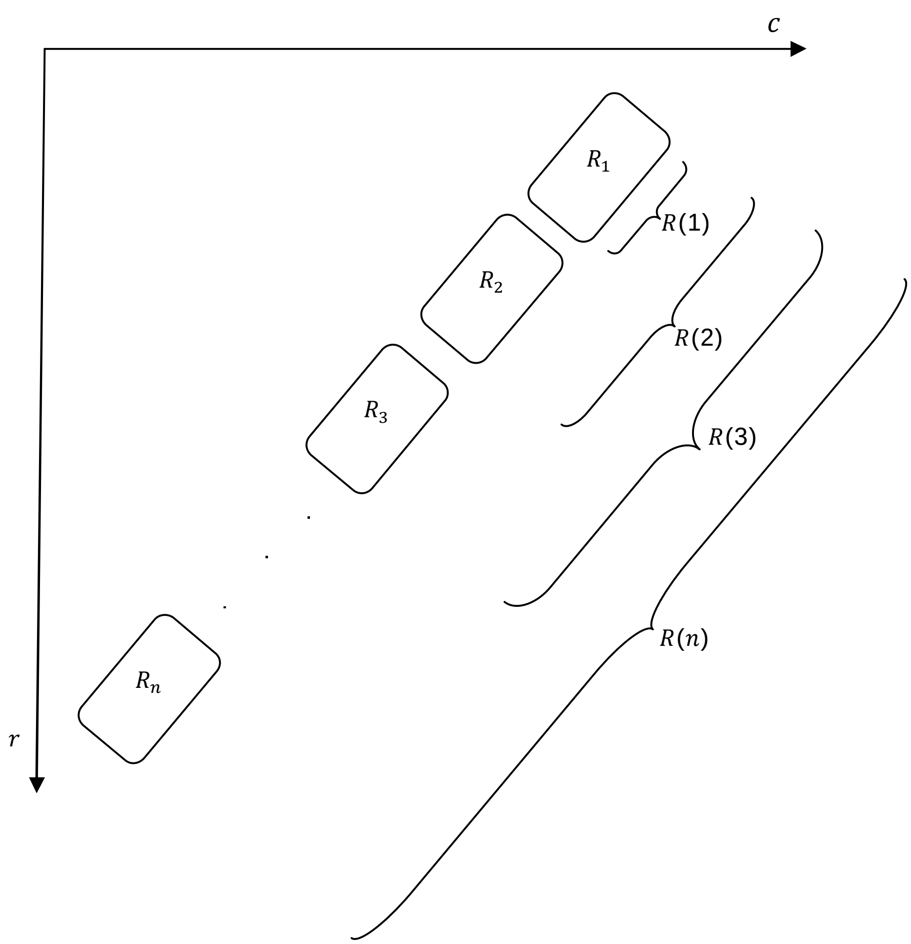

Figure 1: A Representation of n Objects in the Row-Column Space.

Denoted α, the axis of least inertia of a 2-dimensional object, R, is given by:

$$2\alpha = \tan^{-1}\frac{2\mu_{rc}}{\mu_{cc}-\mu_{rr}} \quad \mathbf{1}$$

where μ_{rc} is the second order mixed moment of object R about its centroid, $(\bar{r},\bar{c})$. With A denoting the area of R, μ_{rc} is itself defined according to:

$$\mu_{rc} = \frac{1}{A}\sum_R (r-\bar{r})(c-\bar{c}) \quad \mathbf{2}$$

Further, μ_{rr} is the second order row moment of object R about its centroid, $(\bar{r},\bar{c})$, and is defined according to:

$$\mu_{rr} = \frac{1}{A}\sum_R (r-\bar{r})(r-\bar{r}) \quad \mathbf{3}$$

Also, μ_{cc} is the second order column moment of object R about its centroid, $(\bar{r},\bar{c})$, and is defined according to:

$$\mu_{cc} = \frac{1}{A}\sum_R (c-\bar{c})(c-\bar{c}) \quad \mathbf{4}$$

Although it does not appear in the formula for finding α in Equation 1, it is pertinent to state the formula for the first order row moment, μ_r, of object R about its centroid, $(\bar{r},\bar{c})$:

$$\mu_r = \frac{1}{A}\sum_R (r-\bar{r}) \quad \mathbf{5}$$

Also, it is relevant, to give the formula for the first order column moment, μ_c, of R about $(\bar{r},\bar{c})$:

$$\mu_c = \frac{1}{A}\sum_R (c-\bar{c}) \quad \mathbf{6}$$

In the above, the left hand side of each of Equations 2 through 6 is scaled by A, the area of object R. For our purposes, we find it extremely useful to introduce terminology and notation which allows us to give corresponding Equations (to Equations 2 through 6) whose left hand sides are not scaled by A. Let us call f_{rc} the ***free*** second order mixed moment of object R about its centroid, $(\bar{r},\bar{c})$, and define it according to:

$$f_{rc} = A\mu_{rc} \quad \mathbf{7}$$

So that:

$$f_{rc} = \sum_R (r-\bar{r})(c-\bar{c}) \quad \mathbf{8}$$

In the same vein, we call f_{rr} the ***free*** second order row moment of object R about its centroid $(\bar{r},\bar{c})$, and define it according to:

$$f_{rr} = A\mu_{rr} \quad \mathbf{9}$$

implying:

$$f_{rr} = \sum_R (r-\bar{r})(r-\bar{r}) \quad \textbf{10}$$

By analogy with Equations 7 through 10, the "names" and definitions of f_{cc}, f_r and f_c should be obvious: for brevity, we will not spell them out explicitly. To proceed, we should furnish more information about our notation. We do this in the pictorial context afforded by Figure 1. We denote the i-th object in the figure by R_i; its centroid by $(\bar{r}_i, \bar{c}_i)$; its area by A_i; its second order mixed moment about any arbitrary point (r,c) by $\mu_{rc,i}(r,c)$; its second order row moment about point (r,c) by $\mu_{rr,i}(r,c)$; its second order column moment about (r,c) by $\mu_{cc,i}(r,c)$; its first order row moment about (r,c) by $\mu_{r,i}(r,c)$; and, finally, its first order column moment about (r,c) by $\mu_{c,i}(r,c)$; Similar notation apply to the ***free*** moments, so that, for example, the free second order mixed moment of the i-th object about some point (r,c) is denoted $f_{rc,i}(r,c)$. Further, the first i objects can be viewed as a single object denoted $R(i)$ (see Figure 1). We denote the centroid of $R(i)$ by $(\bar{r}(i), \bar{c}(i))$; its second order mixed moment about any arbitrary point, (r,c), by $\mu_{rc}(i;r,c)$; its second order column moment about (r,c) by $\mu_{cc}(i;r,c)$, its second order row moment about (r,c) by $\mu_{rr}(i;r,c)$,); its first order row moment about (r,c) by $\mu_r(i;r,c)$; and, finally, its first order column moment about (r,c) by $\mu_{c,i}(r,c)$. Again, analogous definitions apply to the free moments, so that, for instance, the free second order mixed moment of $R(i)$ is denoted $f_{rc}(i;r,c)$.

The nature of the problem at hand lends itself to a solution via some sort of recurrence procedure. Indeed, we have found one such recurrence procedure. Looking back at Equation 1, we see that there are three building blocks for computing α: μ_{rc}, μ_{cc}, and μ_{rr}. Our recurrence procedure is in terms of these three building blocks. In other words, the recurrence procedure actually comprises three smaller recurrence procedures: for computing μ_{rc}, μ_{cc}, and μ_{rr} respectively. It turns out that these three smaller recurrence procedures are more or less analogs of one another. Therefore, it suffices to describe just one of them. We choose to describe the one for computing μ_{rc}.

With the foregoing in mind, let us proceed along the line dictated by the statement of our problem. We should begin by computing $\mu_{rc,1}(\bar{r}_1, \bar{c}_1)$, the second order mixed moment of the first object, R_1, about its centroid, $(\bar{r}_1, \bar{c}_1)$:

$$\mu_{rc,1}(\bar{r}_1, \bar{c}_1) = \frac{1}{A_1} \sum_{R_1} (r-\bar{r}_1)(c-\bar{c}_1) \quad \textbf{11}$$

The above Equation can be expressed in terms of the free second order mixed moment, $f_{rc,1}(\bar{r}_1, \bar{c}_1)$:

$$\mu_{rc,1}(\bar{r}_1, \bar{c}_1) = \frac{1}{A_1} f_{rc,1}(\bar{r}_1, \bar{c}_1) \quad \textbf{12}$$

where

$$f_{rc,1}(\bar{r}_1, \bar{c}_1) = \sum_{R_1} (r-\bar{r}_1)(c-\bar{c}_1) \quad \textbf{13}$$

But, we can view the object R_1 as a composite object, $R(1)$, comprising a single object. This allows us to write $(\bar{r}_1, \bar{c}_1) = (\bar{r}(1), \bar{c}(1))$, $\mu_{rc,1}(\bar{r}_1, \bar{c}_1) = \mu_{rc}(1; \bar{r}(1), \bar{c}(1))$, $f_{rc,1}(\bar{r}_1, \bar{c}_1) = f_{rc,1}(\bar{r}(1), \bar{c}(1))$, and $A_1 = A(1)$, so that Equation 12 can be written as:

$$\mu_{rc}(1; \bar{r}(1), \bar{c}(1)) = \frac{1}{A(1)} f_{rc,1}(\bar{r}(1), \bar{c}(1)) \quad \textbf{14}$$

Next, we should compute $\mu_{rc}(2; \overline{r}(2), \bar{c}(2))$, the second order mixed moment of the first two objects, $R(2)$, about the centroid, $(\bar{r}(2), \bar{c}(2))$, of $R(2)$:

$$\mu_{rc}(2; \overline{r}(2), \bar{c}(2)) = \frac{1}{A(2)} \sum_{R(2)} (r - \bar{r}(2)) (c - \bar{c}(2)) \quad \textbf{15}$$

The computation in Equation 15 above is done over $R(2)$ which comprises the objects R_1 and R_2. This is shown explicitly as follows:

$$\mu_{rc}(2; \overline{r}(2), \bar{c}(2)) = \frac{1}{A(2)} \sum_{R_1} (r - \bar{r}(2))(c - \bar{c}(2)) + \frac{1}{A(2)} \sum_{R_2} (r - \bar{r}(2))(c - \bar{c}(2)) \quad \textbf{16}$$

Now, here comes the issue. To compute $\mu_{rc}(1; \overline{r}(1), \bar{c}(1))$, we "went over all the pixels in object R_1 (see Equations 11 through 14); " to compute $\mu_{rc}(2; \overline{r}(2), \bar{c}(2))$, according to Equation 16, we must go over all the pixels in R_1 and R_2, repeating "going over R_1" in the process; by extension, to compute $\mu_{rc}(3; \overline{r}(3), \bar{c}(3))$, we would need to go over R_1, R_2 and R_3, repeating "going over R_1 and R_2" in the process; and so on. Clearly, there is quite a lot of redundancy in the computations described above. The redundancy becomes even more unacceptable when the number of objects gets quite large. For instance, to compute $\mu_{rc}(10; \overline{r}(10), \bar{c}(10))$, we would have to "go over" ten objects, the first nine of which would have been gone over during the computation of $\mu_{rc}(9; \overline{r}(9), \bar{c}(9))$! The question we pose is: ***"Is it possible to harness the computation of*** $\mu_{rc}(i-1; \overline{r}(i-1), \bar{c}(i-1))$ ***towards eliminating the need to go over the first*** $i-1$ ***objects, out of all*** i ***objects, involved in the computation of*** $\mu_{rc}(i; \overline{r}(i), \bar{c}(i))$***."*** As we shall see, the answer is: "Yes, to a large extent."

In presenting our recurrence formula, we shall move from specific to general, and then back to specific again. Let us now proceed by developing the recurrence formula for the computation of $\mu_{rc}(2; \overline{r}(2), \bar{c}(2))$. The centroid of $R(2)$ is $(\bar{r}(2), \bar{c}(2))$, that of R_1 is $(\bar{r_1}, \bar{c_1})$, ***or equivalently*** $(\bar{r}(1), \bar{c}(1))$ (This "***equivalence"*** holds only for R_1). Let us denote the difference between $\bar{r}(2)$ and $\bar{r}(1)$ as Δr_{12}, and write:

$$\Delta r_{12} = \bar{r}(2) - \bar{r}(1)$$

$$\therefore \quad \bar{r}(2) = \bar{r}(1) + \Delta r_{12} \quad \textbf{17}$$

In the same vein, we can denote the difference between $\bar{c}(2)$ and $\bar{c}(1)$ as Δc_{12}, and write:

$$\Delta c_{12} = \bar{c}(2) - \bar{c}(1)$$

$$\therefore \quad \bar{c}(2) = \bar{c}(1) + \Delta c_{12} \quad \textbf{18}$$

Putting the values of $\bar{r}(2)$ and $\bar{c}(2)$, from Equations 17 and 18 respectively, into the first term of Equation 16, we have:

$$\mu_{rc}(2; \overline{r}(2), \bar{c}(2)) = \frac{1}{A(2)} \sum_{R_1} (r - \bar{r}(1) - \Delta r_{12}) (c - \bar{c}(1) - \Delta c_{12})$$

$$+ \frac{1}{A(2)} \sum_{R_2} (r - \bar{r}(2)) (c - \bar{c}(2))$$

$$\mu_{rc}\big(2;\overline{r}(2),\bar{c}(2)\big) = \frac{1}{A(2)}\textstyle\sum_{R_1}(r-\bar{r}(1))\,(c-\bar{c}(1)\,) - \frac{1}{A(2)}\sum_{R_1}\Delta c_{12}(r-\bar{r}(1)\,)$$
$$-\frac{1}{A(2)}\textstyle\sum_{R_1}\Delta r_{12}\,\big(c-\bar{c}(1)\big) + \frac{1}{A(2)}\sum_{R_1}\Delta r_{12}\Delta c_{12} + \frac{1}{A(2)}\sum_{R_2}(r-\bar{r}(2)\,)\,(c-\bar{c}(2)\,) \qquad \mathbf{19}$$

Notice the resemblance of the first term in the Equation above with $f_{rc,1}\,(\overline{r_1},\overline{c_1}\,)$ as defined in Equation 13. Using Equation 13 and the fact that $f_{rc,1}\,(\overline{r_1},\overline{c_1}\,) = f_{rc,1}(\bar{r}(1),\bar{c}(1))$, we have:

$$\mu_{rc}\big(2;\overline{r}(2),\bar{c}(2)\big) = \frac{1}{A(2)}f_{rc,1}(\bar{r}(1),\bar{c}(1)) - \frac{1}{A(2)}\textstyle\sum_{R_1}\Delta c_{12}(r-\overline{r_1}\,) - \frac{1}{A(2)}\sum_{R_1}\Delta r_{12}\,(c-\overline{c_1})$$
$$+\frac{1}{A(2)}\textstyle\sum_{R_1}\Delta r_{12}\Delta c_{12} + \frac{1}{A(2)}\sum_{R_2}(r-\bar{r}(2)\,)\,(c-\bar{c}(2)\,)$$

Similarly, the second and third terms in the above Equation bear clear resemblances with $f_{r,1}(\bar{r}(1),\bar{c}(1))$ and $f_{c,1}\,(\bar{r}(1),\bar{c}(1))$ respectively. This observation lets us to write:

$$\mu_{rc}\big(2;\overline{r}(2),\bar{c}(2)\big) = \frac{1}{A(2)}f_{rc,1}(\bar{r}(1),\bar{c}(1)) - \frac{1}{A(2)}\Delta c_{12}f_{r,1}(\bar{r}(1),\bar{c}(1)) - \frac{1}{A(2)}\Delta r_{12}f_{c,1}\,(\bar{r}(1),\bar{c}(1))$$
$$+\frac{1}{A(2)}\textstyle\sum_{R_1}\Delta r_{12}\Delta c_{12} + \frac{1}{A(2)}\sum_{R_2}(r-\bar{r}(2)\,)\,(c-\bar{c}(2)\,)$$

Next, we factor out all the $\frac{1}{A(2)}$'s, and observe that the summation in the penultimate term of the above is simply $A_1\Delta r_{12}\Delta c_{12}$, where A_1 is the area of R_1. This implies:

$$\mu_{rc}\big(2;\overline{r}(2),\bar{c}(2)\big) = [\,f_{rc,1}(\bar{r}(1),\bar{c}(1)) - \Delta c_{12}f_{r,1}(\bar{r}(1),\bar{c}(1)) - \Delta r_{12}f_{c,1}\,(\bar{r}(1),\bar{c}(1))$$
$$+A_1\Delta r_{12}\Delta c_{12} + \textstyle\sum_{R_2}(r-\bar{r}(2)\,)\,(c-\bar{c}(2)\,)]\,\frac{1}{A(2)} \qquad \mathbf{20}$$

But then, the final term within parentheses in Equation 20 above can be denoted $f_{rc,2}(\bar{r}(2),\bar{c}(2))$. With this, we get:

$$\mu_{rc}\big(2;\overline{r}(2),\bar{c}(2)\big) = [\,f_{rc,1}(\bar{r}(1),\bar{c}(1)) - \Delta c_{12}f_{r,1}(\bar{r}(1),\bar{c}(1)) - \Delta r_{12}f_{c,1}\,(\bar{r}(1),\bar{c}(1))$$
$$+A_1\Delta r_{12}\Delta c_{12} + f_{rc,2}(\bar{r}(2),\bar{c}(2))]\,\frac{1}{A(2)} \qquad \mathbf{21}$$

Equation 21 above is our recurrence formula for the computation of $\mu_{rc}\big(2;\overline{r}(2),\bar{c}(2)\big)$. Towards the end of this write-up, using a case-study, we hope to show how Equation 21 can be more efficient than Equation 15 in the computation of $\mu_{rc}\big(2;\overline{r}(2),\bar{c}(2)\big)$. For now however, let us attempt to obtain a similar recurrence formula for $\mu_{rc}\big(3;\overline{r}(3),\bar{c}(3)\big)$. The plan is to study the formulae for $\mu_{rc}\big(2;\overline{r}(2),\bar{c}(2)\big)$ and $\mu_{rc}\big(3;\overline{r}(3),\bar{c}(3)\big)$ with the aim of generalizing them into a formula for $\mu_{rc}\big(i;\overline{r}(i),\bar{c}(i)\big)$. We begin with:

$$\mu_{rc}\big(3;\overline{r}(3),\bar{c}(3)\big) = \frac{1}{A(3)}\textstyle\sum_{R(3)}(r-\bar{r}(3)\,)\,(c-\bar{c}(3)\,) \qquad \mathbf{22}$$

$$= \frac{1}{A(3)}\textstyle\sum_{R_1}(r-\bar{r}(3)\,)\,(c-\bar{c}(3)\,) + \frac{1}{A(3)}\sum_{R_2}(r-\bar{r}(3)\,)\,(c-\bar{c}(3)\,)$$

$$+\frac{1}{A(3)}\textstyle\sum_{R_3}(r-\bar{r}(3)\,)\,(c-\bar{c}(3)\,) \qquad \mathbf{23}$$

Let Δr_{13} denote the difference between $\bar{r}(3)$ and $\bar{r}(1)$; and Δc_{13} the difference between $\bar{c}(3)$ and $\bar{c}(1)$, then set:

$$\bar{r}(3) = \bar{r}(1) + \Delta r_{13}$$

and $\bar{c}(3) = \bar{c}(1) + \Delta c_{13}$

Substitute the above two Equations for the $\bar{r}(3)$ and $\bar{c}(3)$ which appear under (only) the $\sum_{R_1}$ summation in Equation 23:

$$\mu_{rc}(3;\bar{r}(3),\bar{c}(3)) = \frac{1}{A(3)}\sum_{R_1}(r-\bar{r}(1)-\Delta r_{13})\,(c-\bar{c}(1)-\Delta c_{13}) + \frac{1}{A(3)}\sum_{R_2}(r-\bar{r}(3))\,(c-\bar{c}(3)) + \frac{1}{A(3)}\sum_{R_3}(r-\bar{r}(3))\,(c-\bar{c}(3)) \quad \textbf{24}$$

Next, let Δr_{23} denote the difference between $\bar{r}(3)$ and $\bar{r}(2)$; and Δc_{23} the difference between $\bar{c}(3)$ and $\bar{c}(2)$, then set:

$\bar{r}(3) = \bar{r}(2) + \Delta r_{23}$

and $\bar{c}(3) = \bar{c}(2) + \Delta c_{23}$

Substitute these two Equations into the $\bar{r}(3)$ and $\bar{c}(3)$ under (only) the $\sum_{R_2}$ summation in Equation 24:

$$\mu_{rc}(3;\bar{r}(3),\bar{c}(3)) = \frac{1}{A(3)}\sum_{R_1}(r-\bar{r}(1)-\Delta r_{13})\,(c-\bar{c}(1)-\Delta c_{13}) + \frac{1}{A(3)}\sum_{R_2}(r-\bar{r}(2)-\Delta r_{23})\,(c-\bar{c}(2)-\Delta c_{23}) + \frac{1}{A(3)}\sum_{R_3}(r-\bar{r}(3))\,(c-\bar{c}(3))$$

$$= \frac{1}{A(3)}\sum_{R_1}(r-\bar{r}(1))\,(c-\bar{c}(1)) - \frac{1}{A(3)}\sum_{R_1}\Delta c_{13}(r-\bar{r}(1)) - \frac{1}{A(3)}\sum_{R_1}\Delta r_{13}(c-\bar{c}(1)) + \frac{1}{A(3)}\sum_{R_1}\Delta r_{13}\Delta c_{13} + \frac{1}{A(3)}\sum_{R_2}(r-\bar{r}(2))\,(c-\bar{c}(2)) - \frac{1}{A(3)}\sum_{R_2}\Delta c_{23}(r-\bar{r}(2)) - \frac{1}{A(3)}\sum_{R_2}\Delta r_{23}(c-\bar{c}(2)) + \frac{1}{A(3)}\sum_{R_2}\Delta r_{23}\Delta c_{23} + \frac{1}{A(3)}\sum_{R_3}(r-\bar{r}(3))\,(c-\bar{c}(3))$$

$$\mu_{rc}(3;\bar{r}(3),\bar{c}(3)) = \frac{1}{A(3)}\,f_{rc,1}(\bar{r}(1),\bar{c}(1)) - \frac{1}{A(3)}\Delta c_{13}f_{r,1}(\bar{r}(1),\bar{c}(1)) - \frac{1}{A(3)}\Delta r_{13}f_{c,1}(\bar{r}(1),\bar{c}(1)) + \frac{1}{A(3)}A_1\Delta r_{13}\,\Delta c_{13} + \frac{1}{A(3)}\,f_{rc,2}(\bar{r}(2),\bar{c}(2)) - \frac{1}{A(3)}\Delta c_{23}f_{r,2}(\bar{r}(2),\bar{c}(2)) - \frac{1}{A(3)}\,\Delta r_{23}f_{c,2}(\bar{r}(2),\bar{c}(2)) + \frac{1}{A(3)}A_2\,\Delta r_{23}\,\Delta c_{23} + \frac{1}{A(3)}f_{rc,3}(\bar{r}(3),\bar{c}(3))$$

Rearranging,

$$\mu_{rc}(3;\bar{r}(3),\bar{c}(3)) = \frac{1}{A(3)}\,f_{rc,1}(\bar{r}(1),\bar{c}(1)) + \frac{1}{A(3)}\,f_{rc,2}(\bar{r}(2),\bar{c}(2)) - \frac{1}{A(3)}\Delta c_{13}f_{r,1}(\bar{r}(1),\bar{c}(1)) - \frac{1}{A(3)}\Delta c_{23}f_{r,2}(\bar{r}(2),\bar{c}(2)) - \frac{1}{A(3)}\Delta r_{13}f_{c,1}(\bar{r}(1),\bar{c}(1)) - \frac{1}{A(3)}\,\Delta r_{23}f_{c,2}(\bar{r}(2),\bar{c}(2)) + \frac{1}{A(3)}A_1\Delta r_{13}\,\Delta c_{13} + \frac{1}{A(3)}A_2\,\Delta r_{23}\,\Delta c_{23} + \frac{1}{A(3)}f_{rc,3}(\bar{r}(3),\bar{c}(3))$$

Factoring out $\frac{1}{A(3)}$,

$$\mu_{rc}(3;\overline{r}(3),\bar{c}(3)) = [f_{rc,1}(\bar{r}(1),\bar{c}(1)) + f_{rc,2}(\bar{r}(2),\bar{c}(2)) - \Delta c_{13} f_{r,1}(\bar{r}(1),\bar{c}(1)) - \Delta c_{23} f_{r,2}(\bar{r}(2),\bar{c}(2)) - \Delta r_{13} f_{c,1}(\bar{r}(1),\bar{c}(1)) - \Delta r_{23} f_{c,2}(\bar{r}(2),\bar{c}(2)) + A_1 \Delta r_{13}\, \Delta c_{13} + A_2\, \Delta r_{23}\, \Delta c_{23} + f_{rc,3}(\bar{r}(3),\bar{c}(3))]\frac{1}{A(3)} \quad \textbf{25}$$

Equation 25 above is our recurrence formula for computing $\mu_{rc}(3;\overline{r}(3),\bar{c}(3))$. We wish to generalize Equations 21 and 25 into a formula for computing $\mu_{rc}(i;\overline{r}(i),\bar{c}(i))$. It should not be hard to see that both Equations generalize into an $\mu_{rc}(i;\overline{r}(i),\bar{c}(i))$ given by:

$$\mu_{rc}(i;\overline{r}(i),\bar{c}(i)) = \{ \sum_{k=1}^{i-1} f_{rc,k}(\bar{r}(k),\bar{c}(k)) - \sum_{k=1}^{i-1} \Delta c_{k,i} f_{r,k}(\bar{r}(k),\bar{c}(k)) - \sum_{k=1}^{i-1} \Delta r_{k,i} f_{c,k}(\bar{r}(k),\bar{c}(k)) + \sum_{k=1}^{i-1} A_k\, \Delta r_{k,i}\, \Delta c_{k,i} + f_{rc,i}(\bar{r}(i),\bar{c}(i)) \}\frac{1}{A_i} \quad \textbf{26}$$

From the specifics, we have arrived at the general. Let us now go back to the specific. Using a case study, we wish to show that the recurrence formula (Equation 26) can lead to lesser computations, albeit at the expense of some additional memory. To this end, let us consider a specific case of just two objects, each of which comprises hundred pixels, for a total of a two hundred pixels. Since there are two objects in this case-study, we should compare the direct formula for $\mu_{rc}(2;\overline{r}(2),\bar{c}(2))$, given in Equation 15, with the recurrence formula for $\mu_{rc}(2;\overline{r}(2),\bar{c}(2))$ given in Equation 21:

Direct Method: 200 + 200 subtractions + 200 multiplications + 199 additions + 1 division =
$\mathbf{800}\ \boldsymbol{steps}$

Recurrence Formula *(without "debt"):* 2 subtractions +2 additions + (1 + 1 + 2) multiplications
+ (100 + 100) subtractions + 100 multiplications + 99 additions + 1 division
= $\mathbf{408}\ \boldsymbol{steps}$

Observe the phrase, "without debt," written within parentheses just before the analysis for the recurrence formula. What does this phrase mean? The first thing to note is that the above analysis assumes that the quantities, $f_{rc,1}(\bar{r}(1),\bar{c}(1))$, $f_{r,1}(\bar{r_1},\bar{c_1}))$, and $f_{c,1}(\bar{r_1},\bar{c_1})$ which are needed in the recurrence formula of Equation 21 are available at the time when $\mu_{rc}(2;\overline{r}(2),\bar{c}(2))$ is to be computed. In other words, it does not account for the computation of these three quantities. Yet, the computation of these three quantities must be accounted for, since our statement of problem would ordinarily lead to the assumption that, at the time when $\mu_{rc}(2;\overline{r}(2),\bar{c}(2))$ is being computed, only the quantity $\mu_{rc}(1;\overline{r}(1),\bar{c}(1))$ is available in computer memory. But then, looking at the formula for computing $\mu_{rc}(1;\overline{r}(1),\bar{c}(1))$ in Equation 14, one sees that $f_{rc,1}(\bar{r}(1),\bar{c}(1))$ is an intermediate result towards obtaining $\mu_{rc}(1;\overline{r}(1),\bar{c}(1))$ in that formula. Therefore, if it is known, say, from an IF-ELSE test, that the problem involves more than one object, then while computing $\mu_{rc}(1;\overline{r}(1),\bar{c}(1))$, we can pre-store $f_{rc,1}(\bar{r}(1),\bar{c}(1))$ for use during the computation of $\mu_{rc}(2;\overline{r}(2),\bar{c}(2))$ (and of all other $\mu_{rc}(i;\overline{r}(i),\bar{c}(i))$'s with $i > 2$). This pre-storage incurs no extra computational costs, counting only arithmetic operations. But, on the contrary, as we shall soon see, the computation of $f_{r,1}(\bar{r}(1), \bar{c}(1))$ and $f_{c,1}(\bar{r}(1), \bar{c}(1))$ do incur some additional cost. To see why this is the case, let us begin with the definition of $f_{r,1}(\bar{r}(1), \bar{c}(1))$:

$$f_{r,1}(\bar{r}(1), \bar{c}(1)) = \sum_{R_1} (r - \bar{r}(1))$$

Ordinarily, under the present case-study, the expression above should involve 100 subtractions and 99 additions. But, indeed, Equation 13 reveals that these 100 subtractions are the same ones that show up in the computation of $f_{rc,1}(\bar{r}(1), \bar{c}(1))$, a precursor to $\mu_{rc,1}(\bar{r}(1), \bar{c}(1))$. This means that, during the computation of $f_{rc,1}(\bar{r}(1), \bar{c}(1))$, we can arrange things in such a way that the results of these 100 subtractions are stored in, perhaps, an array of size 100, so that, apart from being available for the computation of $f_{rc,1}(\bar{r}(1), \bar{c}(1))$ itself, they can as well be harnessed for computing $f_{r,1}(\bar{r}(1), \bar{c}(1))$. In summary then, the computation of $f_{r,1}(\bar{r}(1), \bar{c}(1))$ incurs only 99 new computations. The situation for $f_{c,1}(\bar{r}(1), \bar{c}(1))$ is analogous. We first define:

$$f_{c,1}(\bar{r}(1), \bar{c}(1)) = \sum_{R_1} (c - \bar{c}(1))$$

Then, we see that this ordinarily involves 100 subtractions and 99 additions, with the 100 subtractions being the exact ones that already show up in the computation of $f_{rc,1}(\bar{r}(1), \bar{c}(1))$. Therefore, like in the computation of $f_{r,1}(\bar{r}(1), \bar{c}(1))$, the computation of $f_{c,1}(\bar{r}(1), \bar{c}(1))$ incurs only 99 new computations. So, in all, the computation of $f_{r,1}(\bar{r}(1), \bar{c}(1))$ and $f_{c,1}(\bar{r}(1), \bar{c}(1))$ incurs a total of $2 \times 99 = 198$ computations.

Now, these 198 computations are needed towards the computation of $\mu_{rc}(2, \bar{r}(2), \bar{c}(2))$, yet they are actually incurred during the time when $\mu_{rc}(1, \bar{r}(1), \bar{c}(1))$ is being computed. One way to view this situation is to say that $\mu_{rc}(1, \bar{r}(1), \bar{c}(1))$ is incurring some cost on behalf of $\mu_{rc}(2, \bar{r}(2), \bar{c}(2))$. Under this metaphor, this cost is a debt which must ultimately be debited from the account of $\mu_{rc}(2, \bar{r}(2), \bar{c}(2))$. This is the reason we used the phrase "without debts" in the analysis for the recurrence relation above. So, adding the debts now, the result is that a total of $408 + 198 = 606$ computations are involved in the recurrence relation for $\mu_{rc}(2, \bar{r}(2), \bar{c}(2))$. Thus, under the current case-study, we conclude that the recurrence relation allows us to save a total $800 - 606 = 194$ computations. Clearly, this amount of savings is non-trivial, and in fact, even more savings can be expected as the number of objects in the problem increases. The key concept that enabled these savings is computation re-use. We systematically expressed the solution to our problem in terms of re-usable pre-existing building blocks, thereby eliminating redundancies.