A DERIVATION OF A FORMULA IN THE TUTORIAL PAPER OF LAWRENCE RABINER ON HIDDEN MARKOV MODELS.

Abdulrahman O. Ibraheem

Department of Computer Science and Engineering, Obafemi Awolowo University, Ile-Ife, Nigeria.

rahmanoladi@yahoo.com

Statement of Problem: To show the following relation:

$$\sum_{d=1}^{\infty} d\, {a_{ii}}^{d-1}(1 - a_{ii}) = \frac{1}{1 - a_{ii}} \quad \text{where } 0 \leq a_{ii} \leq 1$$

Source of Problem: The above relation occurs as Equation $6b$ in the paper titled "A Tutorial on Hidden Markov Models and Selected Applications to Speech Processing," by Lawrence Rabiner. The paper provides context for the relation.

Solution:

Let us write $S_\infty = \sum_{d=1}^{\infty} d\, {a_{ii}}^{d-1}(1 - a_{ii})$. We wish to show that $S_\infty = \frac{1}{1 - a_{ii}}$

Further, let the sum of the first d terms in the series S_∞ be denoted S_d. We have:

$$S_d = (1 - a_{ii}) + (2\, a_{ii} - 2{a_{ii}}^2) + (3\, {a_{ii}}^2 - 3{a_{ii}}^3) + \cdots + ((d-1)\, {a_{ii}}^{d-2} - (d-1){a_{ii}}^{d-1}) + (d\, {a_{ii}}^{d-1} - d{a_{ii}}^d)$$

$$= 1 + (-a_{ii} + 2a_{ii}) + (-2{a_{ii}}^2 + 3\, {a_{ii}}^2) + \cdots + (-(d-1){a_{ii}}^{d-1} + d\, {a_{ii}}^{d-1}) - d{a_{ii}}^d$$

$$S_d = 1 + a_{ii} + {a_{ii}}^2 + {a_{ii}}^3 + \cdots + {a_{ii}}^{d-1} - d{a_{ii}}^d \qquad \textbf{1}$$

Set $G_d = 1 + a_{ii} + {a_{ii}}^2 + {a_{ii}}^3 + \cdots + {a_{ii}}^{d-1}$ **2**

We can combine Equations 1 and 2 to write:

$$S_d = G_d - d{a_{ii}}^d$$

But, $S_\infty = \lim_{d\to\infty} S_d = \lim_{d\to\infty}(G_d - d{a_{ii}}^d) = \lim_{d\to\infty} G_d - \lim_{d\to\infty} d{a_{ii}}^d$ **3**

Clearly G_d is a geometric series with common ratio $r = a_{ii}$ and first term, $a = 1$. Notice that, because it is given that $0 \leq a_{ii} \leq 1$, it follows that $|r| \leq 1$. The fact that $|r| \leq 1$ allows us to write the sum to infinity of G_d as follows:

$$\lim_{d\to\infty} G_d = \frac{1}{1 - a_{ii}} \qquad \textbf{4}$$

We can put this result in Equation 3 to obtain:

$$S_\infty = \frac{1}{1 - a_{ii}} - \lim_{d\to\infty} d{a_{ii}}^d \qquad \textbf{5}$$

Let us now focus on the term $\lim_{d\to\infty} d{a_{ii}}^{d}$ in Equation 3. We wish to show that $\lim_{d\to\infty} d{a_{ii}}^{d} = 0$. In other words, we wish to show that the sequence $\{ d{a_{ii}}^{d}\}_d$ converges. Define $f(d) = d{a_{ii}}^{d}$, and $f(x) = x{a_{ii}}^{x}$ respectively, where x is a continuous variable which can assume **all values** in the real interval $(0, \infty)$, unlike d which is a discrete variable, which can only assume **positive integer values** on the interval $(0, \infty)$. It should not be hard to see that $f(d)$ is a subsequence of the sequence $f(x)$. A theorem in the Calculus says that if a sequence converges, then its subsequences must also converge. Therefore, to show that $f(d)$ converges, it is sufficient to just show the convergence of $f(x)$.

Now,

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x{a_{ii}}^{x} = \lim_{x\to\infty} \frac{x}{{a_{ii}}^{-x}} = \frac{\infty}{\infty}$, where the third equality rests heavily on the given: $0 \leq a_{ii} \leq 1$

The appearance of the form $\frac{\infty}{\infty}$ suggests that we try L'Hopital's rule. We set $g(x) = x$ and $h(x) = {a_{ii}}^{-x}$, so that :

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{g(x)}{h(x)}$$

L'Hopital's rule allows us to write:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{g'(x)}{h'(x)} = \lim_{x\to\infty} \frac{1}{-{a_{ii}}^{-x} \log {a_{ii}}^{-x}} = 0$$

The above result says that $f(x)$ converges as x tends to infinity. It follows, as earlier discussed, that $f(d)$ too must converge as d tends to infinity. Thus, we have:

$$\lim_{d\to\infty} d{a_{ii}}^{d} = \lim_{d\to\infty} f(d) = 0 \qquad \textbf{6}$$

Putting Equation 6 into Equation 5, we get the result we set out prove:

$$S_\infty = \frac{1}{1 - a_{ii}} - 0 = \frac{1}{1 - a_{ii}}$$